Quantum Computing Solution to Sturm-Liouville Differential Equation

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Abstract

Quantum computing is coming up as futuristic technology in engineering and science. The famous Sturm-Liouville (SL) differential equation along with some specific boundary conditions appear in many branches of physics as well as in other fundamental fields. The numerical solution of such boundary value problems (like heat transfer, potential problem etc.) is frequently computationally expensive, and traditional computers may not be able to deliver accurate solutions in an acceptable length of time. We propose quantum computing approach to solve such boundary value problems that will be more efficient and time solvent. The present research demands that the Harrow-Hassidim-Lloyd (HHL) quantum algorithm provides an efficient technique for solving such boundary value problems that arise in many branches of physical science and engineering. The prime focus of this research is to explore HHL quantum algorithm in detail. The quantum phase estimation algorithm is applied for solving eigen values and eigenvectors associated with the SL type of boundary value problem. As a case study, in this research, we have solved the steady state heat transfer problem with boundary conditions. Our computation is based on Qiskit module made by IBM and available as a library in Python. A detailed analysis of the computational complexity of our algorithm that compares its performance with classical methods on a range of boundary value problems has been discussed. Role of Tridiagonal Toeplitz matrix is revealed. Results demonstrate that the method of quantum algorithm for solving boundary value problems can significantly outperform classical methods, especially for larger problem sizes. This work represents a significant methodology in solving important equations of physics and mathematics in future quantum computers.

Keywords: *Boundary value problems, Quantum computing, HHL algorithm, Quantum phase estimation algorithm*

1. Introduction

Quantum technology is an inevitable future. Almost every branch of physical sciences and engineering, including physics, chemistry, and material sciences are enlightened by the boundary value problems. They involve either ordinary differential equations (ODEs) or partial differential equations (PDEs) with some associated boundary or initial conditions. Understanding and predicting the behavior of physical systems requires the solution of boundary value problems, which is an important topic of research in applied mathematics. For solving boundary value issues, numerical approaches such as finite difference, finite element, and spectral methods have been widely used. [1]**.** These methods have been highly successful in many applications, but they can become computationally expensive and time-consuming for large-scale problems. As a result, there has been growing interest in exploring the potential of quantum computing to solve boundary value problems.

Nevertheless, it is important to note that present hardware support to quantum computing systems [2] has limited qubit resources and high error rates that make it challenging to scale up these algorithms to large scale problems. Moreover, the development of quantum algorithms for boundary value problems is an area of active research and the type of problems that can be efficiently solved on quantum computers needed to be understood yet. Therefore, while quantum computing has the potential to provide a significant speedup for solving certain types of boundary value problems, further research is required to develop empirical and competent algorithms that will run the current as well as future quantum hardware efficiently.

The Harrow-Hassidim-Lloyd (HHL) quantum algorithm [3] is a promising approach for solving system of linear equations that can be applied to certain types of boundary value problems. The HHL algorithm has been shown to provide a polynomial speedup over classical algorithms for solving system of linear equations. Therefore, it has the potential to improve the efficiency of numerical methods for solving boundary value problems significantly. In particular, the HHL algorithm can be used to solve the Poisson equation, which is a common boundary value problem in physics and engineering. Further the HHL algorithm has been proven to provide the so called quantum supremacy for certain types of system of linear equations [4].

An important area of application of the HHL algorithm is quantum physics and quantum chemistry [5] where it can be used to simulate the behaviour of atoms and molecules. In particular, the HHL algorithm can be used to solve the Schrödinger equation, which describes the behaviour of quantum systems. By using the HHL algorithm, quantum chemists can simulate the behaviour of complex molecules and study their properties, potentially leading to the discovery of new materials and drugs. The HHL algorithm can also be applied to optimization problems, such as linear programming and quadratic programming [4]. By using the HHL algorithm, these problems can be solved more efficiently than classical algorithms, potentially leading to faster optimization and improved decision-making.

It is to be noted at this juncture that the HHL algorithm has some limitations, including the need for a large number of qubits and high precision in the input parameters. Moreover, the current limitations of quantum hardware make it challenging to scale up the algorithm to large-scale problems. Therefore, while the HHL algorithm has the potential to provide a significant speedup for solving certain types of boundary value problems, further research is needed to develop practical and efficient algorithms that may be implemented on present and future quantum hardware.

Our paper looks into the potential area of physical problems that can be well understood as the Sturm-Liouville (SL) system and put forward the quantum computational method to solve them efficiently. Sturm-Liouville differential equation has its historical importance in functional analysis for bearing very important role in handling real field problems in pure as well as in applied mathematics. Historically the SL differential equation is the first eigenvalues - eigenfunction problem subjected to some boundary conditions and was first used in classical mechanics to solve for the possible vibrations of a vibrating string fixed at both the ends. Both finite and infinite series solutions were obtained. The spectrum of application of this differential equation is very broad.

The basic goal of this manuscript is to provide a comprehensive overview of the application of the HHL quantum algorithm for solving SL type of boundary value problems. Specifically, we focus on using the HHL algorithm to solve the steady state heat equation which is the most common SL type of boundary value problem in physics and engineering. Using a finite difference method we first discretized the steady state heat equation to procure a system of linear equations. We then implement the HHL algorithm using Qiskit module of IBM under python software to solve these set of equations. Next the efficiency and accuracy of the HHL algorithm is compared with the classical numerical methods for solving the same equation. We also discuss the potential advantages and limitations of the HHL algorithm for solving other types of boundary value problems. We believe that SL type of boundary value problems can be solved more efficiently by using numerical methods following the HHL algorithm. The remaining portion of the document is architecture in the following manner. Sub section 2.1 discusses the properties of Sturm-

Liouville equation as a qualified differential equation where HHL algorithm can be effectively used. 2.2 describes the objectives of the current research. Section 3 presents mathematical background of HHL algorithm. Of which the sub section 3.1 presents the role of quantum phase estimation (QPE) while solving linear system of equations by HHL algorithm. Section 4 describes the statement of the problems under consideration. Section 5 describes the methodology of quantum algorithm HHL for solving a system of linear equations. Section 6 renders the results and discussion and finally section 7 summarizes our study operations by outlining the future scope in light of the aforementioned potential.

2. Mathematical Background

2.1 Quantum version of Sturm-Liouville Differential equation:

The classical version of Sturm-Liouville differential equation deals with the general linear second order boundary value problem. The mathematical expression of the same can be written as

$$\frac{\mathrm{d}}{\mathrm{dx}}\left[\mathrm{k(x)}\frac{\mathrm{dV(x)}}{\mathrm{dx}}\right] + \left(\mathrm{g(x)}\lambda - \mathrm{l(x)}\right)\mathrm{V(x)} = 0, \quad x \in [\alpha, \beta] \tag{1}$$

with the imposed homogeneous boundary conditions

$$k(x)V\acute{(x)}\text{-hV(x)} = 0 \qquad \text{for } x = \alpha \tag{2}$$

$$k(x)V\acute{(x)}\text{+HV(x)} = 0 \qquad \text{for } x = \beta \tag{3}$$

where k, g, l are positive functions, h and H are given positive constants and λ is a parameter.

There are several physical systems that satisfy equation (1). Of them some classical systems that can be mentioned here are a stretched vibrating string clamped at two ends, heat conducting rod with its two ends placed at some heat baths etc. Quantum mechanical systems like particle in one dimensional box can also come within the domain of Sturm-Liouville differential equation. A freely moving quantum particle on a circle is a case of periodic SL system. Other than physical systems as above, it is here to note that Legender equation, Chebyshev equation and Hermite equation shows the property of singular SL system.

Further to note that the Sturm-Liouville equation (1) can be written in the form of eigen value equation

$$£\ \mathrm{V} = \lambda\ \mathrm{V} \tag{4}$$

With the differential Sturm-Liouville operator

$$£ = \frac{1}{g(x)}[-\frac{d}{dx}\left[k(x)\frac{d}{dx}\right] + l(x)] \tag{5}$$

Defined on some domain in the Hilbert space Ή defined by $Ł^2([\alpha,\beta], g(x), dx)$ with square interable functions on $[\alpha,\beta]$ with inner product

$$< p, q > = \int_\alpha^\beta \bar{p}(x)\, q(x) g(x) dx \tag{6}$$

where g(x) is called the weight function. Further Sturm-Liouville operator is hermitian in nature and have a set of complete orthonormal basis in Ή. Therefore, SL equation qualifies to have a quantum solution even for classical system. The HHL algorithm, therefore, can be applied directly to get a quantum computational solution for such problems.

2.2. Research Objectives:

The objective of this manuscript is to explore the potential of the HHL quantum algorithm for solving SL type of boundary value problems. Specifically, our research objectives are:

a) To obtain a system of linear equations corresponding to the differential equation of the BVPs using the finite difference method taking the SL equation into account.

b) To implement the HHL algorithm on a quantum computer to solve the system of linear equations and obtain the solution to the governing SL equation having boundary conditions.
c) To gauge the HHL algorithm's efficiency and accuracy with traditional numerical approaches for solving SL BVPs.
d) To discuss the potential advantages and limitations of the HHL algorithm for solving other types of BVPs.

By achieving these research objectives, we intent to put forward a comprehensive application of the HHL algorithm for solving SL boundary value problems. We believe that our research can contribute to the reinforcement of new quantum algorithms as well as numerical methods for solving complex problems in physics and engineering.

3. Mathematical Background of HHL Algorithm

The HHL (Harrow-Hassidim-Lloyd) algorithm is a quantum mechanical algorithm that has been designed to solve systems of linear equations of the form Ax = p, where A is a Hermitian matrix and p is a vector. The algorithm requires a quantum state preparation routine to prepare an initial state that encodes the matrix A and the vector p. It also requires a quantum subroutine that performs a controlled unitary transformation on the quantum state, conditioned on the values of A and p. Finally, the algorithm requires a quantum measurement routine to measure the solution x. Provided with the Hermitian matrix A and the vector p for a given problem, the algorithm can efficiently find the quantum state |x⟩ that approximates the solution to the linear system of equations

$$Ax = p \tag{7}$$

where the matrix $A \in R^{N \times N}$ and vector $b \in R^N$. The objective is to find $x \in R^N$.Here, the augmented vector $\vec{p}$ will also be an N column vector , where N= 2^n . A and $\vec{p}$ are known according to the given problem and $\vec{x}$ is the unknown to be solved. The target vector $\vec{x}$ which will be thus $\vec{x} = A^{-1}\vec{p}$ also become an N column vector. The very first step towards the application of HHL technique is to encode the equation (7) into the quantum version. Denoting the normalised quantum representation of $\vec{x}$ and $\vec{p}$ as the corresponding ket vectors as $|x> = \sum_j \frac{x_j}{\sqrt{\sum_j x_j^2}}|j>$ and $|p> = \sum_j \frac{p_j}{\sqrt{\sum_j p_j^2}}|j>$, the above equation (7) can be written in the Hilbert space as

$$\boldsymbol{\mathcal{A}}|x>=|p> \tag{8}$$

where $\boldsymbol{\mathcal{A}}$ is the quantum counterpart of matrix A. Equation (8) is therefore eligible to be solved quantum computationally. One of the numerical technique for solving Eq. (8) is an eigen representation named as spectral decomposition (SD). The SD of $\boldsymbol{\mathcal{A}}$ into its eigen basis can be written as

$$\boldsymbol{\mathcal{A}} = \sum_{j=1}^{N} \lambda_j |v_j >< v_j| \tag{9}$$

with $\lambda_j \in R$ and(λ_j, v_j) being the jth eigen pair of $\boldsymbol{\mathcal{A}}$.

We, therefore, can write $\boldsymbol{\mathcal{A}}^{-1} = \sum_{j=1}^{N} \lambda_j^{-1} |v_j >< v_j|$ being $\boldsymbol{\mathcal{A}}$ is Hermitian therefore invertable and its basises are orthogonal. $|b>$ can be written in terms of the basises of $\boldsymbol{\mathcal{A}}$ as

$|b>= \sum_{j=1}^{N} \alpha_j |v_j>$ where $\alpha_j \in C$ and the required solution therefore

$$|x>= \boldsymbol{\mathcal{A}}^{-1}|b>= \sum_{j=1}^{N} \lambda_j^{-1} \alpha_j |v_j> \tag{10}$$

An important factor to be noted here for matrix inversion algorithm is that the matrix $\boldsymbol{\mathcal{A}}$ should not be an "ill conditioned" i.e., the condition number β should follow $\beta^{-2} I \leq \boldsymbol{\mathcal{A}}^{\dagger}\boldsymbol{\mathcal{A}} \leq I$.

Moreover, the demand is also that in case $\boldsymbol{\mathcal{A}}$ is non Hermitian, we can always make the corresponding Hermitian operator as follows,

$$C = \begin{pmatrix} 0 & A \\ A^{\dagger} & 0 \end{pmatrix} \tag{11}$$

Here C is Hermitian, we can solve the equation $C\vec{y} = \begin{pmatrix} \vec{p} \\ 0 \end{pmatrix}$ to obtain $\vec{y} = \begin{pmatrix} 0 \\ \vec{x} \end{pmatrix}$ and follow the same method of solving the system.

The HHL algorithm works by encoding the matrix A and the vector p within a quantum state |ψ⟩, and then accomplishing a quantum circuit that performs a controlled unitary transformation U on the quantum state. The following relation defines the unitary transformation U :

$$U = e^{iAt}[|0\rangle\langle 0| + |1\rangle\langle 1|] \tag{12}$$

where t is a parameter that depends on the condition number of A. The controlled unitary transformation U acts on the first qubits of the quantum state |ψ⟩, and is conditioned on the state of an ancilla qubit. The ancilla qubit is used to control the sign of the eigenvalues of A.

After applying the unitary transformation U, the quantum state is measured and post-processed using classical algorithms to obtain an estimate of the solution |x⟩. The post-processing step involves measuring the second qubits of the quantum state |ψ⟩ using the results of classical linear inversion algorithm that revels an estimation of |x⟩. It is worth mentioning that solving specific types of systems of linear equations, the HHL technique has been proven to yield a polynomial speedup over conventional approaches. The algorithm, however, has some drawbacks, including the requirement for a large number of qubits and high precision in the input parameters.

3.1. Impact of Quantum Phase Estimation (QPE)

Quantum phase estimation (QPE) is a quantum algorithm[6] that estimates the eigenvalues of a unitary matrix. As in many quantum algorithms the HHL algorithm[7] too require QPE as a crucial subroutine. The eigenvalues are then used to determine the value of the parameter t in the unitary transformation U. The precision of the eigenvalue estimation has a direct impact on the accuracy of the HHL algorithm solution. The number of qubits required for QPE depends on the number of eigenvalues to be estimated and the required precision. QPE can be realized with a wide range of quantum circuits, including iterative phase estimation and quantum signal processing methods. However, QPE is a critical component of the HHL algorithm[8] and how accurately it is been estimated directly affect the performance of the HHL algorithm. By exploring the impact of QPE on the HHL algorithm, we can gain insights into the potential advantages and limitations of the algorithm for solving other types of boundary value problems. Overall, the impact of QPE on our manuscript highlights the importance of developing efficient and accurate quantum algorithms for estimating eigenvalues and performing other types of quantum signal processing[9].We can unlock the promise of quantum computing for solving a wide range of issues in physics, engineering, and other domains by enhancing the state-of-the-art in quantum signal processing[10].

4. Statement of the Problem

As case study of Sturm-Liouville differential equation we have chosen the steady state heat equation. The exact mathematical formulation of our case is represented by

$$\frac{d^2\theta}{dx^2} - 0.15\,\theta = 0 \tag{13}$$

With reference to equation (1), taking $k(x) = 1$ and with boundary conditions: $\theta(0) = \theta_1 = 240\ and\ \theta(10) = \theta_6 = 150$, our target is to solve Eq. (13) for temperature distribution along the length of copper wire using classical Thomas algorithm and HHL quantum algorithm.

5. Methodology of solving Ax = p using quantum algorithm

The HHL algorithm being a quantum algorithm that solves a system of linear equations of the form Ax = p, A being a Hermitian matrix, and p a vector, it works by using quantum linear algebra to transform the problem into a form that can be solved efficiently using a quantum computer. Algorithm may be exhibited into the following steps.

The first step of the HHL algorithm is to prepare the quantum state |p⟩/‖p‖, where ‖p‖ is the Euclidean norm of p. This can be done using quantum amplitude amplification or other techniques.

The second step is to transfigure the problem into a form so that a quantum computer can solve it most efficiently. This is done by applying the quantum phase estimation algorithm (QPE) to the matrix A. QPE is a well-known quantum algorithm that can be used for estimating the eigenvalues of a unitary matrix. In the case of the HHL algorithm too the QPE algorithm is employed to estimate the eigenvalues of A.

In third step a unitary transformation is performed on the quantum state and the final state is obtained that contains the solution to the linear system. A set of quantum gates that depend on the eigenvalues of A and the vector |p⟩ does this operation.

Finally, the solution to the linear system is obtained by measuring the quantum state and post-processing the results.

In summary, the algorithm relies on the quantum phase estimation algorithm to estimate the eigenvalues of A and a set of quantum gates transform the quantum state into the solution to the linear system. While the HHL algorithm has the potential to solve systems of linear equations exponentially faster than classical algorithms, practical implementations are still in their infancy due to the challenges of building large-scale quantum computers. However, we also have solved the problem in hand using Thomas algorithm for a comparison of classical solver vs quantum solver.

Numerical solution of problem: At first, we discretize the governing Eq. (13) using central scheme of finite difference method and we obtain the approximation as

$$\frac{(\theta_{i-1}-2\theta_i+\theta_{i+1})}{(\Delta x)^2} - 0.15\theta_i = 0 \tag{14}$$

The domain of interest is here ranged from x = 0 to x = 10 and we discretize the domain by 6 grid points. Though authors are aware of the fact that the increase in the grid points increases the accuracy, but for the sake of demonstration only 6 grid points are chosen. The step size used in the computation is estimated as

$$\Delta x = \frac{x_6 - x_1}{N-1} = \frac{10-0}{5} = 2 \tag{15}$$

So, we have now $x_1 = 0, x_2 = x_1 + \Delta x = 2, x_3 = 4, x_4 = 6, x_5 = 8, x_6 = 10$. Therefore, the temperature vector results $[\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6]^T$. It is obvious that we need to compute $\theta_2, \theta_3, \theta_4$ and θ_5 because θ_1 and θ_6 are boundary conditions as mentioned. Discretization of boundary conditions results $\theta_1 = 240$ and $\theta_6 = 150$. Using Eq. (15) into Eq. (14) we finally obtain

$$\theta_{i-1} - 2.6\,\theta_i + \theta_{i+1} = 0 \tag{16}$$

Substituting values of index, i as 2, 3, 4 and 5, into Eq. (17), we obtain the following system of linear equations

$$\begin{gathered}
\theta_1 - 2.6\,\theta_2 + \theta_3 = 0 \\
\theta_2 - 2.6\,\theta_3 + \theta_4 = 0 \\
\theta_3 - 2.6\,\theta_4 + \theta_5 = 0 \\
\theta_4 - 2.6\,\theta_5 + \theta_6 = 0
\end{gathered} \tag{17}$$

After plugging in the boundary conditions, we can write the matrix representation of system of equations (18) as

$$\begin{bmatrix} -2.6 & 1 & 0 & 0 \\ 1 & -2.6 & 1 & 0 \\ 0 & 1 & -2.6 & 1 \\ 0 & 0 & 1 & -2.6 \end{bmatrix} \begin{bmatrix} \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix} = \begin{bmatrix} -240 \\ 0 \\ 0 \\ -150 \end{bmatrix} \quad (18)$$

We have applied Thomas algorithm for solving Eq. (18) classically. However, as per our research interest we have applied HHL quantum algorithm to solve system of Eq. (18) too. The function tridiagonaltoeplitz() is applied for qubit mode of quantum algorithm HHL implementation.

6. Results and discussion

We have solved the above mentioned problem both by Thomas algorithm as well as HHL quantum algorithm.

6.1. Results of the above Problem

According to the problem statement above, the HHL algorithm is employed with the following inputs:

Domain of interest: $0 \leq x \leq 10$

Number of grid points taken into consideration is 6

Left boundary point of x = 0 and right boundary point of x = 10

Temperature of the left boundary point is kept at 240ºC

Temperature of the right boundary point is kept at 150ºC.

To solve the heat equation we have used the traditional explicit finite difference method as classical one and the quantum computing method follows the HHL quantum algorithm. Results are shown in Table 1.

Quantum computing results of the temperature of the system using HHL algorithm are structured as

- **`state`**: either as a vector or as a circuit that prepares the solution
- **`euclidean_norm`**: The Euclidean norm is calculated by the algorithm if contains the knowledge to calculate it.
- **`observable`**: the computed observable(s) list
- **`circuit_results`**: the observable results from the (list of) circuit(s)

First, our result is focussed towards`"classical solution" at inner mesh points as an array` [118.11, 67.09, 56.33, 79.36] which is obtained by using a classical three point central difference algorithm, and corresponding states are represented in the form of an array by calling state as classical state: [0.41732882 0.23705664 0.19901844 0.2803913].

Other two stages of the operation uses quantum algorithms. Hence we may only ingress into the quantum state which is achieved by yielding the quantum circuits as shown in Fig. 1 and Fig. 2. The quantum circuits prepare the solution state as (i) naïve state (Fig. 1) and (ii) tridiagonal state (Fig. 2).

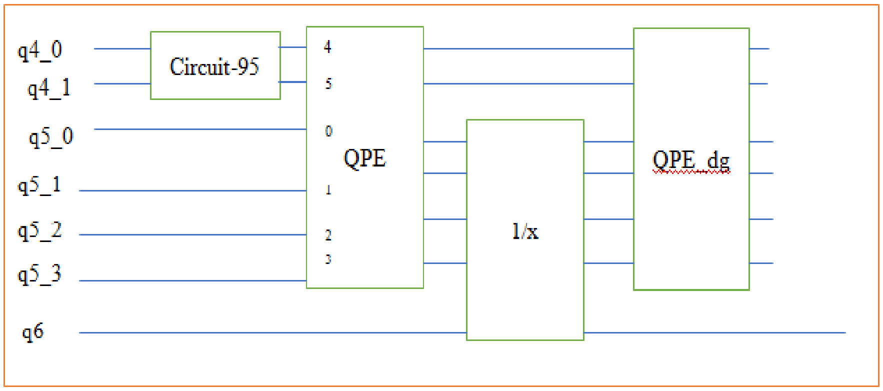

FIG 1.Naïve State

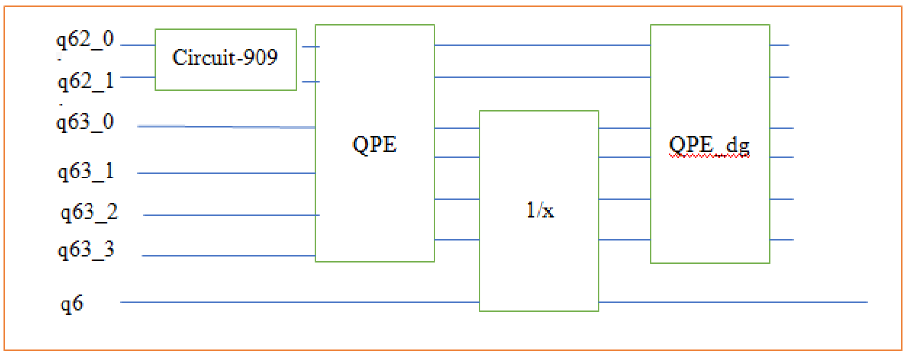

FIG 2. Tridiagonal State

Traditional classical method of finite difference yields the solution at inner mesh points as: Classical solution of temperature: [118.11 67.09 56.32 79.35]

Remembering the definition of the Euclidean norm for a vector, the probability of measuring in the auxiliary qubit when the same operation is performed in a quantum device is the vector's squared norm. This implies that the HHL algorithm can always be used to calculate the solution's euclidean norm, allowing for the comparison of the results' accuracy. Thus execution of this step yields the Classical Euclidean norm as 0.5904, Naive Euclidean norm as 0.5941 and Tridiagonal Euclidean norm as 0.5938.

We also produce the raw solution vectors as:

Naive raw solution vector: [[$-4.74990998e^{-15}$+0.02024402j 1.68586977e^{-16}-0.01286636j]]

Tridiagonal raw solution vector: [-3.45e-7 + 0.02017j 4.805e-5 -0.013j]

Real part of full naive solution vector:[-0.59377125 0.02107453]

Full tridiagonal solution vector : [-0.00426 0.5938]

Classical state: [0.41732882 0.23705664 0.19901844 0.2803913]

Table 1: **Temperature at the inner points of the system by Classical and Quantum Method**

Value of x (m)	Temperature by NumPy linear Solver method (°C)	Temperature by Tridiagonal Method (°C)	Absolute Error
0	240	240	0

2.0	117.99	118.11	0.01
4.0	66.77	67.09	0.32
6.0	55.93	56.32	0.39
8.0	79.17	79.35	0.18
10.0	150	150	0

Graphical representation of solution of Eq. (13) using classical FDM (Thomas Algorithm) and quantum FDM (HHL algorithm) are shown in Fig. 3 and Fig. 4. Table 2 presents the depth of quantum solution, depth of tridiagonal solution and excess corresponding to size of the system 2×2, 4×4, 8×8 and 16×16 respectively.

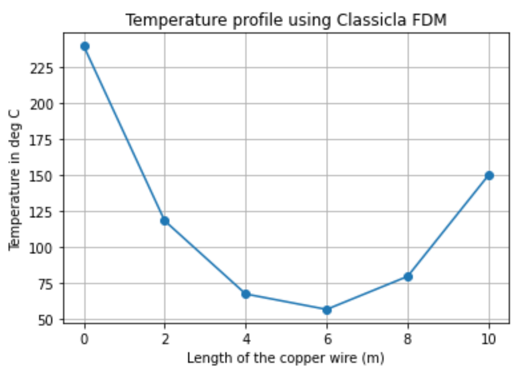

Fig.3. Temperature profile by classical FDM

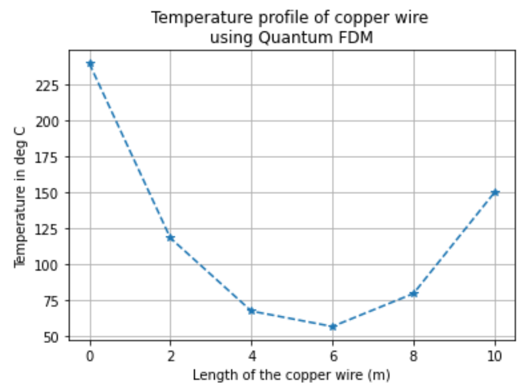

Fig.4. Temperature profile by quantum FDM

Table 2: **Properties of Quantum and Classical solution with the size of the system**

Size of the System	2×2	4×4	8×8	16×16
Quantum solution depth	335	2592	32924	349593
Tridiagonal solution depth	629	3427	14756	51832
Excess	-294	-835	18168	297761

From Table 2, it can be easily told that with the increase of size of the system, the quantum method scales up exponentially in the number of qubits whereas Tridiagonal Toeplitz increases polynomially. The terms "classical functional" and "quantum functional" refer to the mathematical operations performed by classical and quantum algorithms, respectively, to transform the input data into the form required for solving the problem in the context of the HHL algorithm for solving a linear system of equations obtained in the domain of numerical method. The quantum functional refers to the quantum circuit used in the HHL algorithm to encode the input matrix A and vector p into a quantum state. The quantum circuit entails applying quantum gates and measurements to a group of qubits in order to convert the input data into a quantum state that can be utilized to solve the linear problem efficiently. On the other hand, the classical functional refers to the classical algorithm used to prepare the input data for the HHL algorithm. This may involve classical computations such as matrix-vector multiplication, and is typically much less efficient than the quantum functional used in the HHL algorithm. The interpretation of the quantum circuit results in the HHL algorithm depends on the specific implementation of the algorithm and the problem being solved. In general, the output of the HHL algorithm is a quantum state that encodes the solution to the linear system of equations. This quantum state can be measured to obtain an estimate of the solution to the problem, which can then be converted into a classical result.

The accuracy and efficiency of the HHL algorithm depend on several factors, including the size and structure of the input matrix, the number of qubits used in the quantum circuit, and the quality of the quantum gates and measurements. The interpretation of the quantum circuit results therefore requires an understanding of the specific implementation of the HHL algorithm being used. Having this concept, we have also computed the value of quantum functional as 0.313348 (other way of interpreting it as probability of occurrence of the transformation from input data to quantum data) and the value of classical functional as 0.31225 (probability of being transformed into the classical state) which implies that the quantum solution does not violate the classical solution other than speed of computation which is exponentially faster than classical computation. However, it is important to note that the output of a quantum circuit, including the quantum functional in the HHL algorithm, is typically probabilistic in nature. This means that the output may be a superposition of different states with certain probabilities, and measuring the output will collapse the quantum state to a single classical result with some probability. The probability distribution of the final classical result can provide information about the accuracy and precision of the solution obtained using the HHL algorithm.

The quantum circuit is interpreted by their corresponding numerical results in the form of an array [(0.604+0j), (0.499+0j), (0.104+0j)].

7. Conclusion

The paper presents a quantum computational approach to solve SL based boundary value problems. The authors show that this approach has the potential to provide consequential speedup over classical methods for certain types of boundary value problems. The approach is based on using the HHL algorithm to solve the linear system of equations that arises by discretizing the boundary value problem. A comprehensive description of the methodology and analysis to represent its computational complexity are discussed. The paper also includes a numerical example that demonstrates the potential of the approach. The authors show that the HHL algorithm can solve a two-point boundary value problem with a tridiagonal Toeplitz matrix exponentially faster than classical algorithms.

The 'excess' qubits in the HHL algorithm are typically used to implement a phase estimation procedure required to estimate the eigenvalues of the input matrix. The eigenvalues can then be used to compute the inverse of the matrix,. The use of excess qubits in the HHL algorithm allows for a more efficient and accurate solution to the linear system. However, the use of excess qubits also increases the complexity and resource requirements of the algorithm as more qubits are required for the quantum circuit.

The prime constraint of the present system is that the given matrix is to be Hermitian. In future our methodology will be focussed towards the quantum computing of SL equation where matrix will be non-Herimitian.

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